MECHANICAL ENGINEERING DEPARTMENT
UNITED STATES NAVAL ACADEMY

EM423 - INTRODUCTION TO MECHANICAL VIBRATIONS

CONTINUOUS SYSTEMS - FLEXURAL VIBRATION OF BEAMS
PART 1: WAVE SOLUTION

SYMBOLS

r	Mass density
A_x	Cross sectional area
M	Bending moment
V	Internal shear force
I	Second moment of area
x	Distance along the beam
k	Wave number

INTRODUCTION

This theory is applicable to the flexural vibration of slender beams. The theory is directly applicable to many engineering structures. The solution to beam vibrations can be derived in two different forms; as a classical solution of equations, or as a wave equation. Part 1 of these notes (this handout) derives the wave solution. The solution using the classical approach is in Part 2.

ASSUMPTIONS

1. The beam is thin compared to its length.

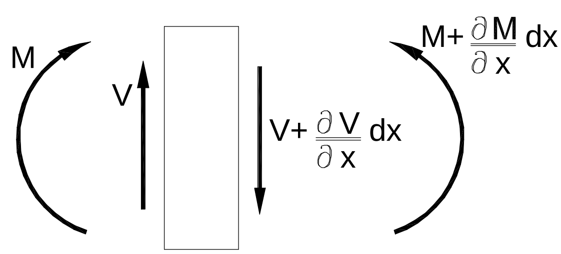
2. The beam is uniform, homogeneous and isotropic.

3. The material is within the elastic limit, and obeys Hooke's Law.

4. Plane sections remain plane.

THEORY PART 1 - THE WAVE SOLUTION

Consider a small element of beam:



We will consider the "strengths" equations, and two "dynamics" equations, and use these three equations to eliminate the internal shear force and bending moment.

<u>Resolve forces vertically (f = ma)</u>

$$\left(V + \frac{\partial V}{\partial x} dx\right) - V = \left(r A_x dx\right) \frac{d^2 y}{dt^2}$$

hence

$$\frac{\partial V}{\partial x} = r A_x \frac{d^2 y}{dt^2}$$

<u>The bending moment equation from Strength of Materials is:</u>

$$M = -EI \frac{\partial^2 y}{\partial x^2}$$

<u>For the element, take moments about the right face and equate Iα:</u> Note we assume that the $I\alpha$ term is small, which is normally valid if we do not have a deep-section beam.

$$\left(M + \frac{\partial M}{\partial x} dx\right) - M - V dx = 0$$

$$V = \frac{\partial M}{\partial x}$$

Combining these equations yields the equation of motion:

$$\boxed{EI \frac{\partial^4 y}{\partial x^4} + r A_x \frac{d^2 y}{dt^2} = 0}$$

Remember, this equation assumes the beam is uniform along its length.

We now restrict the motion to be harmonic with respect to time, and separate the variables. This is a 4th order equation, and the general solution could include functions such as sine and cosine, as well as the hyperbolic functions sinh and cosh. More generally, for the wave solution discussed in this handout, we use the complex exponential form, which can be combined to produce the sin, cos, sinh, and cosh functions if necessary.

$$y(x,t) = Y(x) e^{iwt}$$

Substitute this solution into the equation of motion to get:

$$\boxed{\begin{array}{l} \dfrac{\partial^4 Y}{\partial x^4} - \left(\dfrac{r A_x w^2}{EI}\right) Y = 0 \\ \text{or} \qquad \dfrac{\partial^4 Y}{\partial x^4} - k^4 Y = 0 \quad \text{with} \quad k = \left(\dfrac{r A_x w^2}{EI}\right)^{1/4} = \text{ wave number} \end{array}}$$

Substituting $Y = A.e^{sx}$ yields $\left(s^4 - k^4\right)Y = 0$ from which:

$$s = \pm k \quad \pm \text{ik}$$

and the general solution for flexural motion of the beam becomes:

$$\boxed{y(x,t) = \left(A_1.e^{kx} + A_2.e^{-kx} + A_3.e^{ikx} + A_4.e^{-ikx}\right)e^{iwt}}$$

<u>**The 3rd term.**</u> Let us inspect this result one term at a time, starting with the 3rd term.

$$A_3.e^{i(wt+kx)}$$

Remembering Euler's equations:

$$e^{\pm kx} = \cosh(kx) \pm \sinh(kx)$$
$$e^{\pm ikx} = \cos(kx) \pm i.\sin(kx)$$

We can see that the 3rd term of the general solution, $A_3\, e^{i(wt+kx)}$, is harmonic with respect to both time and position along the beam. As time increases, we can only keep the value of the term constant if we decrease x. This means the term represents <u>a wave traveling in the negative x direction.</u>

The wave number is k, from which we can calculate the speed of the flexural waves, a_f

$$a_f = f.l_f = \frac{w}{k} \qquad \text{hence} \qquad a_f^2 = w\sqrt{\frac{EI}{rA_x}}$$

<u>Unlike strings, rods and shafts,</u> the wave velocity depends on its frequency. Waves with this property are called DISPERSIVE.

As the wave travels down the beam, the beam has associated kinetic and potential (spring-like) energies. These energies are:

$$\text{KE/unit length} = \frac{1}{2} r A_X \left|A_3\right|^2 w^2 \sin^2\left(wt - kx + q\right)$$
$$\text{PE/unit length} = \frac{1}{2} r A_X \left|A_3\right|^2 w^2 \cos^2\left(wt - kx + q\right)$$

The total energy is a constant.

$$\text{Total energy per unit length = KE+ PE} = \frac{1}{2} r A_X w^2 \left|A_3\right|^2$$

The power flow associated with this wave can be calculated as:

$$\begin{aligned}\text{Power flow} &= \frac{\text{energy}}{\text{unit length}} \times \text{speed} \\ &= \frac{1}{2} r A_X w^2 \left|A_3\right|^2 a_f \\ &= \frac{1}{2} r A_X \frac{w^3}{k} \left|A_3\right|^2\end{aligned}$$

<u>The 4th Term.</u> The 4th term in the solution is:

$$A_4.e^{i(wt-kx)}$$

This is similar to the 3rd term, and represents <u>a wave traveling in the positive x direction.</u> It is harmonic with respect to both time and position, and has the same speed as the 3rd term wave.

<u>The 2nd term.</u> Now let us inspect the 2nd term.

$$A_2.e^{-kx}e^{iwt}$$

The constants *A_1, A_2, A_3 and A_4* may themselves be complex. As an alternative way of inspecting the second term, let us make the substitution:

$$A_2 = \left|A_2\right| e^{if_2}$$

which makes the second term

$$\left|A_2\right| e^{-kx} e^{i(wt+f_2)}$$

The (negative) real exponent shows the motion is NOT HARMONIC WITH RESPECT TO POSITION along the beam. In other words, for a given instant in time, the entire wave has the same sign. The motion is in phase for all *x*, and the negative exponential term shows the motion decays in the positive *x*-direction. Waves with this property are called non propagating waves.

The complex exponential term shows the motion is harmonic with respect to time, as we expect from our initial assumptions.

The 1st term. The 1^{st} term is similar to the 2^{nd} term, but the wave decays in the negative *x*-direction.

NON PROPAGATING WAVES decay with distance from the point where they are generated. Hence they cannot be seen at large distances from where they are generated. Because of this, they are also sometimes called near field waves.

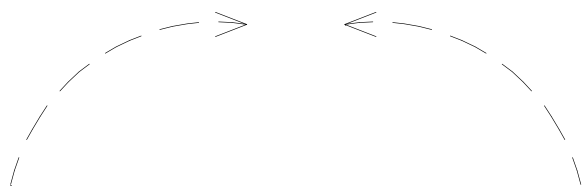
PROPAGATING WAVES have a constant amplitude (ignoring any energy loss), and travel a long way through the structure. They are also called far field waves.

DISCUSSION

1. Harmonic flexural vibration of a beam includes 4 waves.

2. There is one far field wave traveling in the positive x-direction, and one in the negative x-direction.

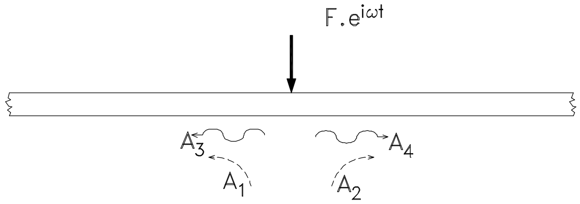
3. There is one near field wave traveling in the positive x-direction, and one in the negative x-direction.



4. Far field waves travel throughout a structure. Near field waves decay rapidly with distance from their point of generation.

5. Far field waves can transmit energy. Near field waves hold energy, but they cannot transmit it through a structure.

EXAMPLE What is the wave motion generated by a point harmonic force in an infinitely long beam?



$$y^- = \left(A_1.e^{+kx} + A_3.e^{+ikx}\right)e^{iwt} \qquad\qquad y^+ = \left(A_2.e^{-kx} + A_4.e^{-ikx}\right)e^{iwt}$$

The boundary conditions for this problem are defined at x = 0 (the point of application of the force).

At x = 0

Displacement continuity: $y^- = y^+$ hence

$$A_1 + A_3 = A_2 + A_4$$

Slope continuity: $\dfrac{\partial y^-}{\partial x} = \dfrac{\partial y^+}{\partial x}$ hence

$$A_1 + i.A_3 = -A_2 - i.A_4$$

Moment balance: $-EI\dfrac{\partial^2 y^-}{\partial x^2} = \dfrac{\partial^2 y^+}{\partial x^2}$ hence

$$A_1 - A_3 = A_2 - A_4$$

Force balance: $F.e^{iwt} = EI\left(\dfrac{\partial^3 y^+}{\partial x^3} - \dfrac{\partial^3 y^-}{\partial x^3}\right)$ hence

$$\frac{-F}{EIk^3} = A_1 - i.A_3 + A_2 - i.A_4$$

Simultaneous solution of these 4 equations yields:

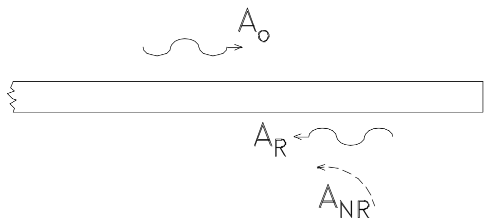
$$A_1 = A_2 = \frac{-F}{4EIk^3} \quad \text{and} \quad A_3 = A_4 = \frac{-i.F}{4EIk^3}$$

From which:

$$y^+ = \frac{-F}{4EIk^3}\left(e^{-kx} + i.e^{-ikx}\right)e^{iwt} \quad \text{and} \quad y^- = \frac{-F}{4EIk^3}\left(e^{kx} + i.e^{ikx}\right)e^{iwt}$$

EFFECT OF A FREE BOUNDARY

We wish to investigate what happens when a far field wave is traveling in a semi-infinite beam, and it hits the end of the beam. The input wave, with amplitude A_O, is the only wave traveling in the positive x-direction. When it hits the boundary, it will be reflected. There is the possibility of two waves being generated at the boundary; one far field, A_R, and one near field, A_{NR}. Pictorially, the wave situation is:



The total motion is a combination of the incident and reflected waves, and therefore only the relevant terms are included in the solution.

The origin for x (x = 0) is conveniently placed at the boundary (right end), with x being positive to the right. In this case the total wave motion is:

$$y = A_O e^{i(wt-kx)} + A_R e^{i(wt+kx)} + A_{NR} e^{iwt} e^{kx}$$

We solve this equation for the boundary conditions. For a free end the boundary conditions are zero bending moment and zero shear force.

$$EI \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{hence} \quad -k^2 A_O - k^2 A_R + k^2 A_{NR} = 0$$

$$EI \frac{\partial^3 y}{\partial x^3} = 0 \quad \text{hence} \quad i.k^3 A_O - i.k^3 A_R + k^3 A_{NR} = 0$$

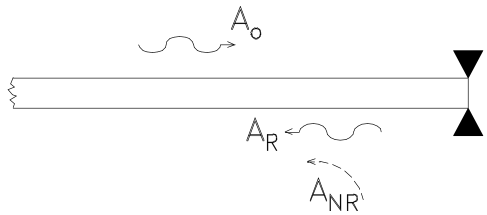
Solving these two equations yields:

$$A_R = -i.A_O \quad \text{and} \quad A_{NR} = (1-i) A_O$$

The effect of the boundary is to introduce a -π/2 phase change to the far field wave. In addition, a near field wave is generated.

EFFECT OF A PINNED BOUNDARY

We now investigate the effect on the far field wave if the boundary is a pin joint. As with the previous example, there is the only wave traveling in the positive x direction, and there is the possibility of two waves being generated at the boundary; one far field and one near field. Pictorially, the wave situation is:



Again, only the relevant terms are included in the solution to give the total motion. As before, the origin for *x* is placed at the boundary, so:

$$y = A_O e^{i(wt-kx)} + A_R e^{i(wt+kx)} + A_{NR} e^{iwt} e^{kx}$$

The boundary conditions for a pin joint are zero displacement and zero bending moment, so solving the general solution at *x* = 0 yields:

$$y = 0 \quad \text{hence} \quad A_O + A_R + A_{NR} = 0$$

$$-EI\frac{\partial^2 y}{\partial x^2} = 0 \quad \text{hence} \quad -k^2 A_O - k^2 A_R + k^2 A_{NR} = 0$$

Solving these two equations yields:

$$A_R = -A_O \quad \text{and} \quad A_{NR} = 0$$

The effect of the boundary is to introduce a 180^o phase change to the far field wave, but no near field wave is generated.

EFFECTS OF DISCONTINUITIES

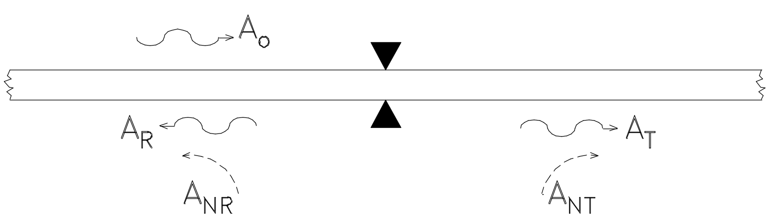
A discontinuity is anything that affects the progress of waves down a beam. Examples of discontinuities are:

A change in the geometry of the beam (e.g. thickness or shape) that changes the second moment of area.

A change in material or material property (e.g. steel to aluminum).

A physical support, such as the frames in a ship or airplane structure.

This handout uses the example of a pin joint to show how to calculate the resulting wave motion. A far field wave travels up to the discontinuity. Some of the wave may reflect, some may be transmitted, and near field waves may be generated. The resulting wave motion is therefore:



In the same way as we used superposition to look at the wave motion in the semi-infinite beams, we use superposition to look separately at the motion on each side of the discontinuity:

$$y^{+}=\left(A_T e^{-ikx}+A_{NT}e^{-kx}\right)e^{iwt}$$

$$y^{-}=\left(A_O e^{-ikx}+A_R e^{ikx}+A_{NR}e^{kx}\right)e^{iwt}$$

There are four unknowns, so we require four boundary conditions. For the pin joint these are displacement and slope continuity and a moment balance.

at $x=0$ $\quad y^{+}=0$ *hence* $A_T+A_{NT}=0$

$y^{-}=0$ *hence* $A_O+A_R+A_{NR}=0$

$$\left(\frac{\partial y^{+}}{\partial x}\right)=\left(\frac{\partial y^{-}}{\partial x}\right) \quad \text{hence} \quad -i.A_T-A_{NT}=-i.A_O+i.A_R+A_{NR}$$

$$\left(\frac{\partial^2 y^{+}}{\partial x^2}\right)=\left(\frac{\partial^2 y^{-}}{\partial x^2}\right) \quad \text{hence} \quad -A_T+A_{NT}=-A_O-A_R+A_{NR}$$

Solution of these four simultaneous equations yields:

$$A_R = \frac{-A_O\left(1+i\right)}{2} \qquad A_T = \frac{A_O\left(1-i\right)}{2}$$
$$A_{NR} = \frac{-A_O\left(1-i\right)}{2} \qquad A_{NT} = \frac{-A_O\left(1-i\right)}{2}$$

These results show that the amplitude of the reflected and transmitted waves is the same, with a -135°/-45° phase shift respectively.

TRANSMISSION AND REFLECTION COEFFICIENTS

The far field waves not only cause motion, but also allow energy to flow through the structure (Recall that the near field waves hold energy, but cannot transfer energy). The power flowing in a far-field wave is given by:

$$\text{Power flow} = \frac{1}{2} r A_X \frac{w^3}{k} \left|A_3\right|^2$$

When the discontinuity does not change the beam's mass per unit length, rA_X, or wave number, the power flow is proportional to (amplitude)^2, and we can define a reflection coefficient and a transmission coefficient.

$$\text{Reflection coefficient } = a_R = \frac{\left(\text{reflected wave energy}\right)}{\left(\text{incident wave energy}\right)} = \frac{\left|A_R\right|^2}{\left|A_O\right|^2}$$

$$\text{Transmission coefficient } = a_T = \frac{\left(\text{transmitted wave energy}\right)}{\left(\text{incident wave energy}\right)} = \frac{\left|A_T\right|^2}{\left|A_O\right|^2}$$

For the pin joint discontinuity, these coefficients are both 0.5. If there is no energy loss at the discontinuity (as assumed in these notes), then

$$a_R + a_T = 1$$

If the beam's geometry or material change, we need to use the full power flow equations in the reflection and transmission coefficient equations.

SUMMARY OF COMMON BOUNDARY CONDITIONS

	DISPLACEMENT	SLOPE	BENDING MOMENT	SHEAR FORCE
	y	$EI\frac{\partial y}{\partial x}$	$EI\frac{\partial^2 y}{\partial x^2}$	$EI\frac{\partial^3 y}{\partial x^3}$
FIXED	zero	zero		
FREE			zero	zero
PINNED	zero		zero	
SLIDER		zero		zero

ASSIGNMENTS

1. Find the resulting wave motion when a far field wave of amplitude A_O traveling in a beam reflects at a fixed boundary.

2. An infinitely long beam has, at x = 0, a pin joint. Find the resulting wave motion when a far field wave of amplitude A_O traveling in the beam hits the discontinuity.

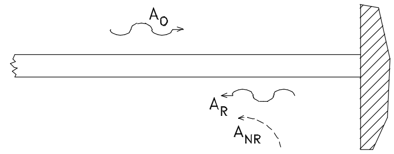
3. How is the wave motion for the previous example changed if the discontinuity is a slider?

4. In the previous two problems, what proportion of the incident wave energy is transmitted along the beam?

5. (Extra credit) A beam of constant cross sectional shape and area is made of steel at the left end, and brass at the right end. What proportion of the incident wave energy is transmitted along the beam when a far field wave in the steel meets the material discontinuity, and what proportion is reflected?

SOLUTIONS

1. Find the resulting wave motion when a far field wave of amplitude A_O traveling in a beam reflects at a fixed boundary.



$$y(x,t) = \left(A_O e^{-ikx} + A_R e^{ikx} + A_{NR} e^{kx}\right) e^{iwt}$$

$$\frac{\partial y}{\partial x} = \left(-ikA_O e^{-ikx} + ikA_R e^{ikx} + kA_{NR} e^{kx}\right) e^{iwt}$$

Boundary conditions at *x = 0*

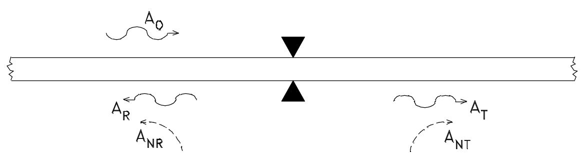
$$y = 0 \quad \text{so} \quad A_O + A_R + A_{NR} = 0$$

$$\frac{\partial y}{\partial x} = 0 \quad \text{so} \quad -iA_O + iA_R + A_{NR} = 0$$

Solve simultaneously to get:

$$A_R = -iA_O \quad \text{and} \quad A_{NR} = -(1-i)A_O$$

2. An infinitely long beam has, at x = 0, a pin joint. Find the resulting wave motion when a far field wave of amplitude A_O traveling in the beam hits the discontinuity.



$$y^-(x,t) = \left(A_O e^{-ikx} + A_R e^{ikx} + A_{NR} e^{kx}\right) e^{iwt} \qquad y^+(x,t) = \left(A_T e^{-ikx} + A_{NT} e^{-kx}\right) e^{iwt}$$

Boundary conditions at *x = 0*

$$y^+ = 0 \quad \text{so} \quad A_T + A_{NT} = 0$$

$$y^- = 0 \quad \text{so} \quad A_O + A_R + A_{NR} = 0$$

$$\left(\frac{\partial y^-}{\partial x}\right) = \left(\frac{\partial y^+}{\partial x}\right) \quad \text{so} \quad -iA_O + iA_R + A_{NR} = -iA_T + A_{NT}$$

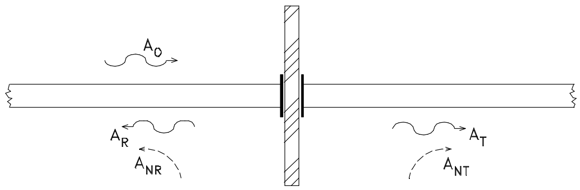
$$\left(\frac{\partial^2 y^-}{\partial x^2}\right) = \left(\frac{\partial^2 y^+}{\partial x^2}\right) \quad \text{so} \quad -A_O - A_R + A_{NR} = -A_T + A_{NT}$$

Solve simultaneously to get:

$$A_R = \frac{-A_O}{(1-i)} \qquad A_T = \frac{-iA_O}{(1-i)}$$

$$A_{NR} = \frac{iA_O}{(1-i)} \qquad A_{NT} = \frac{-iA_O}{(1-i)}$$

3. How is the wave motion for the previous example changed if the discontinuity is a slider?



Boundary conditions at $x = 0$

$$y^- = y^+ \qquad \text{so} \qquad A_O + A_R + A_{NR} = A_T + A_{NT}$$

$$\left(\frac{\partial y^-}{\partial x}\right) = 0 \qquad \text{so} \qquad -iA_O + iA_R + A_{NR} = 0$$

$$\left(\frac{\partial y^+}{\partial x}\right) = 0 \qquad \text{so} \qquad -iA_T + A_{NT} = 0$$

$$\left(\frac{\partial^3 y^-}{\partial x^3}\right) = \left(\frac{\partial^3 y^+}{\partial x^3}\right) \qquad \text{so} \qquad iA_O - iA_R + A_{NR} = iA_T + A_{NT}$$

Solve simultaneously to get:

$$A_R = \frac{A_O(1-i)}{2} \qquad A_T = \frac{A_O(1+i)}{2}$$

$$A_{NR} = ? \qquad A_{NT} = ?$$

4. In the previous two problems, what proportion of the incident wave energy is transmitted along the beam?

$$(\text{Transmitted Energy}) \propto (\text{Amplitude})^2$$

For both cases $\left|\frac{A_T^2}{A_O^2}\right| = 0.5$, so 50% of the incident energy is transmitted, and 50% is reflected.

5. (Extra credit) A beam of constant cross sectional shape and area is made of steel at the left end, and brass at the right end. What proportion of the incident wave energy is transmitted along the beam when a far field wave in the steel meets the material discontinuity, and what proportion is reflected?

The solution to extra credit problems is available from your instructor.